CHAPTER 10 – EXPANDING OUR NUMBER SYSTEM

SIGNED NUMBERS, OPERATIONS, AND NUMBER SYSTEMS

10.1 Ways of Thinking About Signed Numbers

Positive and negative numbers are often called "signed numbers" because of the + or – signs that are used. The + sign is normally omitted.

DISCUSSION

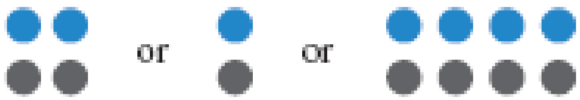
How could each of the following be used to think about signed numbers? Describe what positive and negative numbers, and zero, would mean.

1. Financial matters like bank balances, profit/loss, paycheck/bill, income/debt, credit cards, and so on.
2. Temperature changes
3. Sea levels
4. Sports settings like football and golf
5. Diets
6. Atomic charges (although atomic charges may not be part of the K−6 curriculum)
7. Games in which you can "go in the hole"

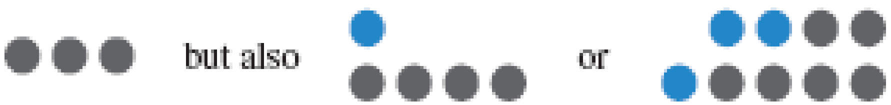
We will focus now on two other ways to represent signed numbers: chips of two colors and the number line. Chips of two colors are an adaptation of the ancient Chinese method of 200 B.C. We will use blue for positive and gray for negative. For example, three blue chips can represent $^{+}3$ (or 3), and 4 gray chips can represent $^{-}4$. Of course, any two colors can be used, so long as it is clear which represents positive numbers and which negative numbers.

The way we arrange these chips together can ultimately represent numbers as well…

Each of these three drawings is a way of showing 0:



And these are the same too…



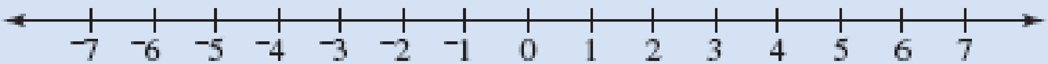
Think About . . .

What are several ways of showing $^{+}2$ with the chips, besides with two blue chips? Why are these ways more abstract than showing just two blue chips?

Think About . . .

What is the opposite of 2? What is the opposite of $^{-}2$? What is the opposite of the opposite of 2? What is the opposite of the opposite of $^{-}2$? What is the opposite of 0? Is ^{-}a always a negative number? What are the different meanings of the $-$ sign in $5 - {}^{-}({}^{-}2)$?

When we combine the set of whole numbers with their opposites, including zero, we obtain a set of numbers we call **integers**. That is, $I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$. These numbers can be represented on the number line as follows:



We say that 6 and $^{-}6$ are ***opposites*** or **additive inverses** of one another because their sum is 0. Generally speaking, the opposite of a is also denoted by ^{-}a, whether a is positive or negative, and $a + {}^{-}a = 0$.

Summary: What does the "-" mean?

- It takes on different meaning depending on the context
  - Much like how a fraction could mean a part/whole, could signify a ratio, could represent a percent, could specify part/part,etc…
- What are the main conceptualizations of the – sign?
  1. The operation of **subtraction**
  2. Indicating a **negative** number
  3. The **opposite**
- What do we typically refer to the – sign as in each case?
  1. The minus sign
  2. The negative sign
  3. ?

Although -6 can be read as “negative six,” “the opposite of six,” or “the additive inverse of six,” -a should be read as “the additive inverse of a” or “the opposite of a,” but not “negative a.” This is because if a itself happens to be negative, then -a is actually positive.

The context usually clarifies the situation. For instance 68 - 24 is subtraction, -5 is the use of “negative,” and -x would refer to an additive inverse.

$$68 - 24$$

$$^{-}5$$

10.6 - Number Systems

The need for different ways to quantify more difficult quantities has led to the introduction of many different “sets” of numbers.

- The **natural numbers** $\mathbb{N}$
  - This is the set of positive whole numbers
  - It is also referred to as the counting numbers
- The **integer numbers** $\mathbb{Z}$
  - This is the set of positive or negative whole numbers
- The **rational numbers** $\mathbb{Q}$
  - This is the set of all fractions $\frac{p}{q}$ consisting of integers
    - *Question*: Does this set include negative values?
    - *Question*: Does this set include decimal values?
    - *Question*: Is every decimal rational?
- The **irrational numbers** $\mathbb{R}\backslash\mathbb{Q}$
  - Contains non-terminating non-repeating decimals
  - Examples: $\pi, e, \sqrt{2}, \ln 2\,, \sqrt[4]{5}, \sin 1\,, \tan 1\,, \cosh 1\,,$ etc.
- The **real numbers** $\mathbb{R}$
  - Includes all of the above
- The **imaginary numbers** $\mathbb{C}\backslash\mathbb{R}$
- The **complex numbers** $\mathbb{C}$

DISCUSSION

1. Think of any two rational numbers, such as $\frac{7}{12}$ and $\frac{13}{15}$. Find another rational number between the two numbers.
2. Find another rational number between $\frac{7}{12}$ and the number you found in part (a). This will give a second number between $\frac{7}{12}$ and $\frac{13}{15}$.
3. How many rational numbers are there between $\frac{7}{12}$ and $\frac{13}{15}$ in all? ■

A set of numbers is **dense** if, for every choice of two different numbers from the set, there is always another number from the set that is between them (the ***density property***).

Think About . . .

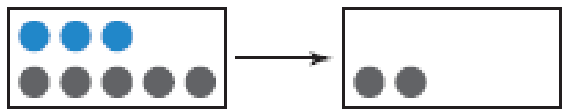
How does the density property assure that there are infinitely many rational numbers, not just one, between every two different rational numbers?

Just as there are several ways to think about signed numbers, there are several ways to think about adding and subtracting them. Combining the colored chips that we discussed in the last section is one way to do this:

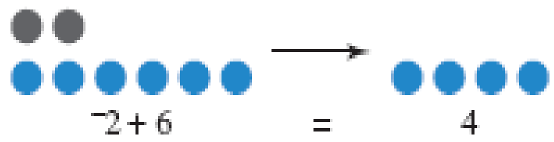
$^{+}3 + {}^{+}2 = n$ $\qquad$ $^{-}4 + {}^{-}3 = n$

$^{+}3 + {}^{+}2 = {}^{+}5$ $\qquad$ $^{-}4 + {}^{-}3 = {}^{-}7$

EXAMPLE



For $^{-}2 + 6 = n$, two gray chips cancel two of the six blue chips, leaving four blue chips, so $^{-}2 + 6 = 4$.

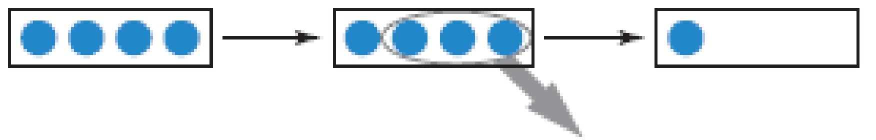


Notice that in effect, when the signs differ, just finding the difference in the numbers of chips for the addends, and then giving that difference the sign of the larger number of chips yield the sum. You may have learned something like that as a rule for adding numbers with different signs.

EXAMPLE

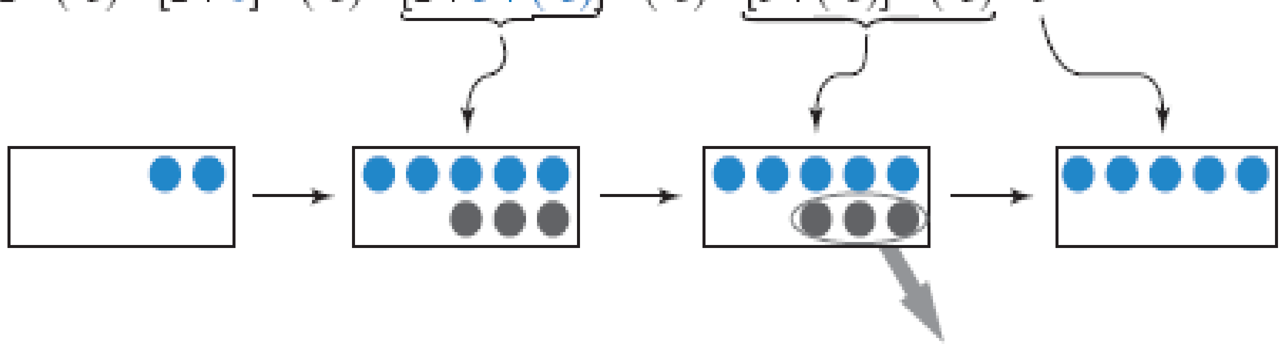
We can use the “take away” model of subtraction to model how to subtract using colored chips:

For example, for $4 - 3 = 1$, we could show the following:



Or, for $^-5 - {}^-2 = {}^-3$,

$$2 - (^-3) = [2 + 0] - (^-3) = [2 + 3 + (^-3)] - (^-3) = [5 + (^-3)] - (^-3) = 5$$



Activity

Use the "combine" and "take-away" model to conceptualize adding and subtracting of integers.

i. $4 + 3$

ii. $4 + (\,^{-}3)$

iii. $^{-}4 + (\,^{-}3)$

iv. $4 - 3$

v. $4 - (\,^{-}3)$

vi. $^{-}4 - (\,^{-}3)$

Question: Which of these are the same?

ACTIVITY

Now, turning our attention to the number line, how could one picture a sort of “hopping on the number line” for each of the following?

a) 6 + 5

b) 3 + 4

c) -6 + 5

d) 6 + -9

e) 7 + -5

f) 6 - 5

g) 6 + -5

h) 11 - (-4)

Special cases:

If $a = 0$, *then* $a + b = b$ and $b + a = b$. We call 0 the **additive identity.**

If $a = {}^-b$, then $a + b = {}^-b + b = 0$ (the additive identity). We call each of b and ${}^-b$ the **additive inverse** of the other because their sum is 0.

Subtraction of signed numbers:

If c and d are signed numbers, then $c - d = c + ({}^-d)$.

Although we have used the take-away view of subtraction with the chips and the number line to motivate the rule for subtraction of signed numbers, the missing-addend view of subtraction could also be used. Recall that the missing-addend approach for $c - d$ would ask: What can be added to d to get c? From $c - d = x$, a symbolic line of reasoning could also result in the $c - d = c + {}^{-}d$ rule, as follows:

$$c - d = x$$

Then, thinking of the missing addend for $c - d$, we get

$$x + d = c$$

$$x + d + {}^{-}d = c + {}^{-}d \quad \text{(by adding } {}^{-}d \text{ to both sides)}$$

$$x + 0 = c + {}^{-}d$$

$$x = c + {}^{-}d$$

So, $c - d = x = c + {}^{-}d$.

ACTIVITY

Can the colored chips and the number line also be used with the missing-addend view of subtraction? Try them with the following:

1. $^{-}4 - {}^{-}2$ (Think: What can be added to $^{-}2$ to get $^{-}4$?)

2. $5 - {}^{-}1$

3. $^{-}6 - 2$

A number's distance from 0 on the number line is called the **absolute value** of the number, and we consider this value to be positive (or zero in the case of zero). We denote the absolute value of a number *b* as $|b|$.

Example:

We can say that $|6|$ is 6, and similarly, $|{}^{-}6|$ is 6. Both 6 and $^{-}6$ are 6 units away from zero. Opposite numbers (always) have the same absolute value.

The properties of whole numbers and rational numbers continue to remain true when negatives and irrational numbers come into play.

For example,

$$-13 + 4 = 4 + -13$$

or

$$(3 + -7) + -5 = 3 + (-7 + -5)$$

Think About . . .

Does the associative property of addition allow you to ignore the parentheses when only addition is involved? Why or why not?

A set of numbers is **closed** under an operation if, when operating on every two numbers in the system, the result is also in the set of numbers.

Example:

When we add any two positive rational numbers, such as $9\frac{3}{4}$ and 5, the sum, $14\frac{3}{4}$, is also a rational number. This property extends to include negative and positive numbers. $^-9\frac{3}{4} + 5 = {}^-4\frac{3}{4}$, a rational number. In both cases, the sum of two rational numbers was another rational number, so we could say that this example illustrates that *the set of rational numbers is closed under addition.*

10.3 Multiplying and Dividing Signed Numbers

You may have heard the rhyme: “Minus times minus is plus, the reason for this we need not discuss.” The reasoning for assigning the sign of the answer when multiplying signed numbers has often been thought of as just using a rule. But here we want to consider this multiplication much more deeply.

Multiplications involving positive numbers or zero are already familiar. The product of two positive numbers is positive, and if zero is a factor, the product is zero. The other cases involve multiplying numbers (1) of opposite signs and (2) when both are negative. We will focus on integers, although the same results will apply to all real numbers.

Earlier we found that one way of thinking about multiplication is as repeated addition. Applying this view of multiplication to $4 \times {}^{-}2$ gives

$$4 \times {}^{-}2 = {}^{-}2 + {}^{-}2 + {}^{-}2 + {}^{-}2 = {}^{-}8$$

So from the discussion on the previous slide, we have found that in general:

(positive) × (negative) = (negative)

But we know that the multiplication of integers must be commutative. For example, $-2 \times 4 = 4 \times -2$. In general then, using the statement at the top, we derive:

(negative) × (positive) = (negative)

Considering a double negative as "opposite" can help in illustrating why $-2 \times -4 = 8$, or in general:

(negative) × (negative) = (positive)

Multiplying two signed numbers.
If the signs of the two numbers are the same, the product will be positive. If the signs of the two numbers are different, the product will be negative.

The ***multiplicative identity*** for the set of rational numbers is 1 because for every rational number a, $1 \cdot a = a$, and $a \cdot 1 = a$.

If the product of two numbers is 1, each number is the **multiplicative inverse** of the other number. If a is not 0, its multiplicative inverse is often written $\frac{1}{a}$ or even a^{-1}. The multiplicative inverse of a (nonzero) fraction is sometimes called its **reciprocal**.

Think About . . .

What is the reciprocal of $\frac{m}{n}$? How does your answer satisfy the description above? Is there a multiplicative identity for the set of integers? Do integers have multiplicative inverses? Why doesn't 0 have a multiplicative inverse?

A second way to show that defining the product of two negative numbers to be positive makes sense mathematically is illustrated in this example, which depends heavily on the distributive property.

Suppose the product of concern is $^-3 \cdot {}^-2$.

Start with $^-3 \cdot 0 = 0$. Substitute $^+2 + {}^-2$ for the first 0.

$$^-3 \cdot (^+2 + {}^-2) = 0 \quad \text{after the substitution.}$$

$$(^-3 \cdot {}^+2) + (^-3 \cdot {}^-2) = 0 \quad \text{using the distributive property.}$$

$$^-6 + (^-3 \cdot {}^-2) = 0 \quad \text{using the known } {}^-3 \cdot {}^+2 = {}^-6.$$

So, $^-3 \cdot {}^-2$ must be equal to $^+6$ to make the equation true.

Considering the division of signed numbers becomes a matter of applying what we've already learned regarding multiplication. Consider...

If a, b, and c are real numbers and b is not 0, then $c \div b = a$ if $a \cdot b = c$.

Example:

a.	$12 \div 4 = 3$	because	$3 \cdot 4 = 12$
b.	$^{-}12 \div 4 = {}^{-}3$	because	$^{-}3 \cdot 4 = {}^{-}12$
c.	$12 \div {}^{-}4 = {}^{-}3$	because	$^{-}3 \cdot {}^{-}4 = 12$
d.	$^{-}12 \div {}^{-}4 = 3$	because	$3 \cdot {}^{-}4 = {}^{-}12$

Multiplication and division of signed numbers:
The product or quotient of two numbers with the same sign is positive. The product or quotient of two numbers with opposite signs is negative.

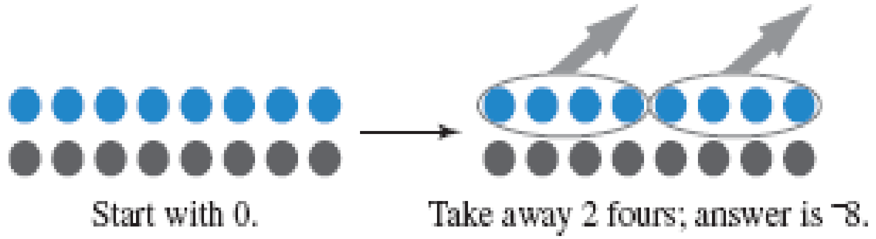
The equality link between $\frac{a}{b}$ and $a \div b$ extends to signed numbers. Hence, $\frac{^-17}{5}$, $\frac{17}{^-5}$, and $^-\left(\frac{17}{5}\right)$ are all equal, because $\frac{^-17}{5} = (^-17) \div 5 = {}^-\left(\frac{17}{5}\right)$ and $\frac{17}{^-5} = 17 \div {}^-5 = {}^-\left(\frac{17}{5}\right)$. This equality between $\frac{a}{b}$ and $a \div b$ leads to a common way of defining the rational numbers in advanced work.

A **rational number** is any number that can be expressed in the form

$$\frac{\text{integer}}{\text{nonzero integer}}.$$

We could also use the colored chips to demonstrate how we handle signed integers in multiplication or division (using repeated addition and subtraction)

For $^{-}2 \times 4$, the work might proceed as follows:



For $^{-}2 \times {}^{-}4$, we would take away two sets each with 4 gray chips, so the answer is the 8 blue chips left, or $^{-}2 \times {}^{-}4 = {}^{+}8$.

Historical Perspective

- By the sixteenth century negative numbers began to appear in algebraic expressions but were treated as fictitious numbers, and were referred to as “false.”
- In 1796, Frend, a Cambridge mathematician, produced an algebra text that completely avoided negative numbers. He said that those who consider multiplying a negative by a negative find their supporters “...amongst those who love to take things upon trust and hate the labor of serious thought.”
- Mathematicians **now** view the use of negative numbers as obvious and a necessity.
- Students too show resistance to negative numbers.
  - But because we use them in regard to temperature, or for debits in finance, we accept their existence often in terms of usefulness in these realms.
  - However, when we begin operating on these numbers, it quickly becomes difficult to justify all the operations.
- With signed numbers, especially in terms of multiplication and division, children are not always able to build their knowledge with intuition.
  - In terms of educating kids regarding some of these more challenging concepts, while sometimes we must learn “rules,” these rules are key to understanding math more deeply