Ch. 2 - Numeration Systems

Chapter 2 – Numeration Systems

- Many people don't know that there are many different ways to represent numbers.
- Historically people have used a variety of **numeration systems** to describe the *value* of a quantity
- Many differences between numeration systems are similar to the differences in languages
- Not all numeration systems work the same way that ours does
- Many symbols are built into numeration systems to expand values, such as $\sqrt{3}$, $\frac{3}{5}$, 0.2, or $8^{1/4}$

2.1 - Ways of Expressing Quantities

- The need to quantify and express values of quantities led to the invention of numeration systems.
- A variety of words and symbols, called **numerals**, have been used to communicate number ideas.
- Our Hindu-Arabic numeration system uses ten digits: 0,1,2,3,4,5,6,7,8,9
- Almost all modern societies use the Hindu-Arabi numeration system.
  - Not everyone uses the same *symbols* though!
  - *Discuss*: German versus English use of period and comma
- With decimal points, fraction bars, and other functions like the square root we can represent almost any number conceivable.

Group Discussion

- Why do you think numerals are used so much in society?
- What are advantages of using special symbols over using just words to express numbers?

Class Activity

Below are different ways of representing twelve.

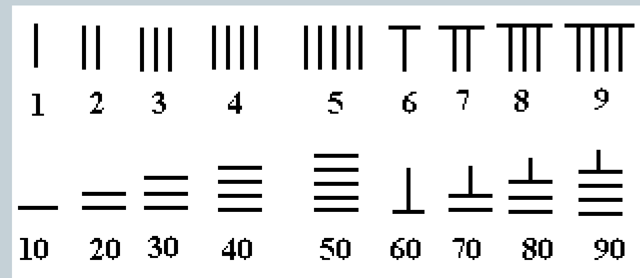
- Can you deduce what each individual mark represents?
- How would the number ten have been written in each case?
- How about eighty-seven? (see next few slides)

Old Greek

1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ϵ	epsilon	50	ν	nu	500	ϕ	phi
6	ϛ	vau*	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	ϙ	koppa*	900	ϡ	sampi

*vau, koppa, and sampi are obsolete characters

Old Chinese



How would we write the following numbers?

- 87
- 206

Roman Numerals

Roman Numeral Table

1	I	14	XIV	27	XXVII	150	CL
2	II	15	XV	28	XXVIII	200	CC
3	III	16	XVI	29	XXIX	300	CCC
4	IV	17	XVII	30	XXX	400	CD
5	V	18	XVIII	31	XXXI	500	D
6	VI	19	XIX	40	XL	600	DC
7	VII	20	XX	50	L	700	DCC
8	VIII	21	XXI	60	LX	800	DCCC
9	IX	22	XXII	70	LXX	900	CM
10	X	23	XXIII	80	LXXX	1000	M
11	XI	24	XXIV	90	XC	1600	MDC
12	XII	25	XXV	100	C	1700	MDCC
13	XIII	26	XXVI	101	CI	1900	MCM

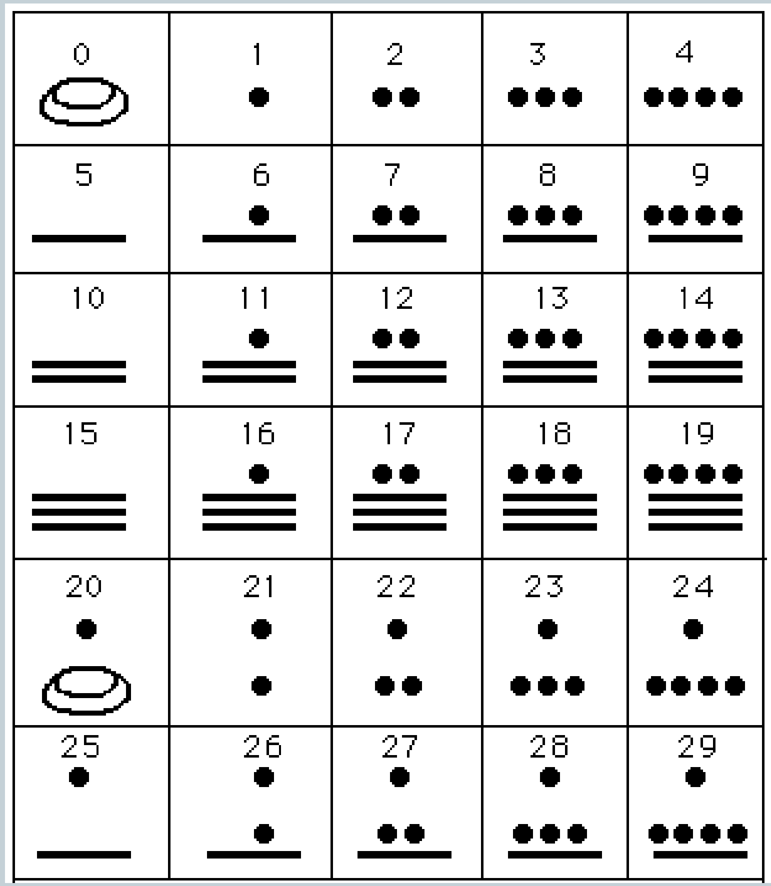
How would we write the following numbers?

- 87
- 206
- 1,294

What does the following stand for?

- LII
- CLXXXV
- MDVII
- MMMDXIX

Mayan


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29

Some ancient cultures did not need many number words. For example, in a recently discovered culture in Papau New Guinea, the same word—“doro”—was used for 2, 3, 4, 19, 20, and 21. By pointing also to different parts of the body and saying “doro,” these people could tell which number was intended.

Think About . . .

Why do you think we use ten digits in our numeration system? Would it make sense to use twenty? Why or why not?

Group Discussion

- Make a list of three quantities to answer the following questions.

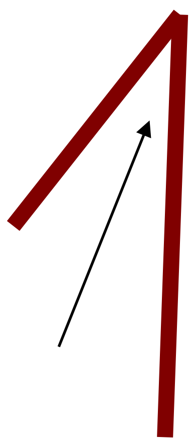
- What quantities, and therefore what number words, would you expect a caveman to have found useful?
  - Assume the caveman knows modern language
- A person in a primitive agricultural society?
- A pioneer or explorer?
- An ordinary modern day civilian?
- An astronaut?
- A mathematician?

Why do we use the symbols 0,1,…9?

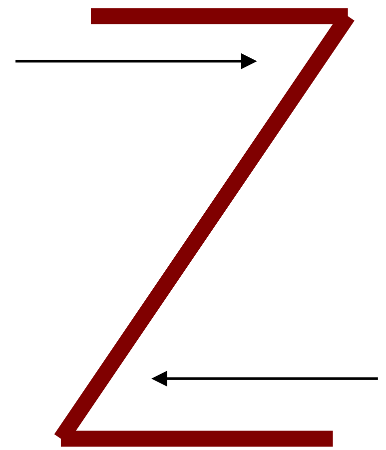
- Why do we use the symbols we use?
  - 0
  - 1
  - 2
  - 3
  - 4
  - 5
  - 6
  - 7
  - 8
  - 9

Why 0,1,2,3…9?

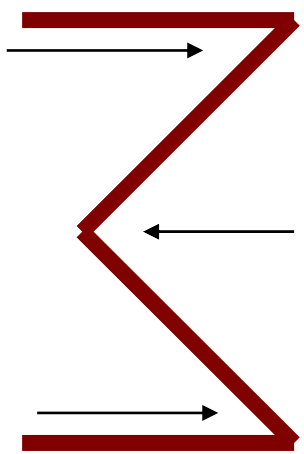
- These numbers descend from Arab algorithms, designed to be easily read and distinguished from Roman numerals.
- The Arabs popularized these algorithms, but the origin goes back to Phoenician merchants that used them to count and for commercial contability (bookkeeping).
- The logic invested in the "Arabic algorithm" is simply the number of angles! (see slides)
  - Note – Implied "less than 180 degrees"



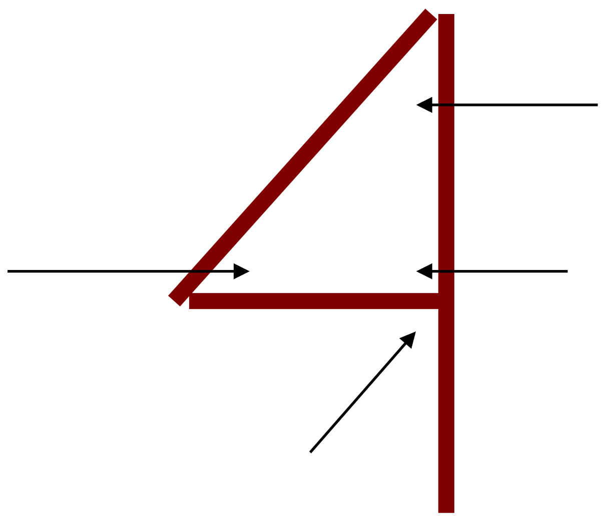
1 angle



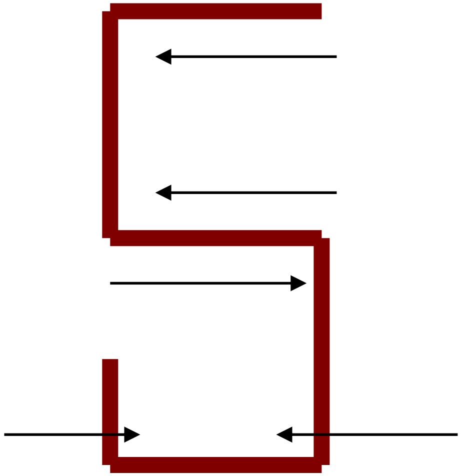
2 angles



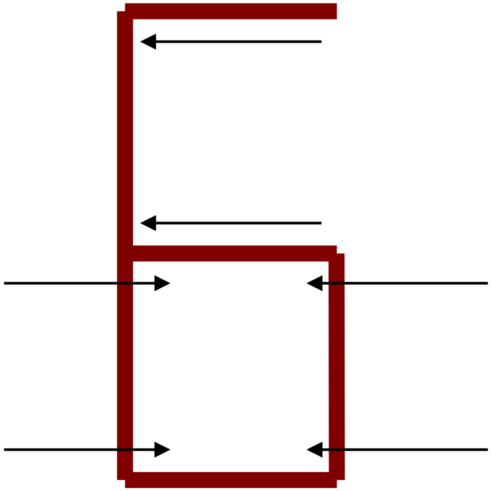
3 angles



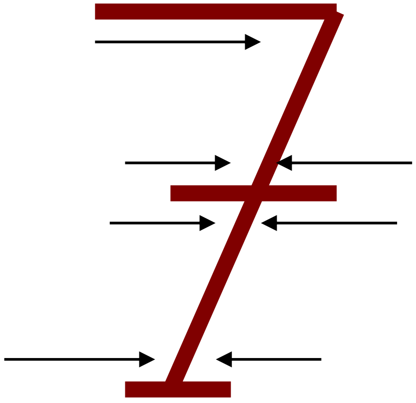
4 angles



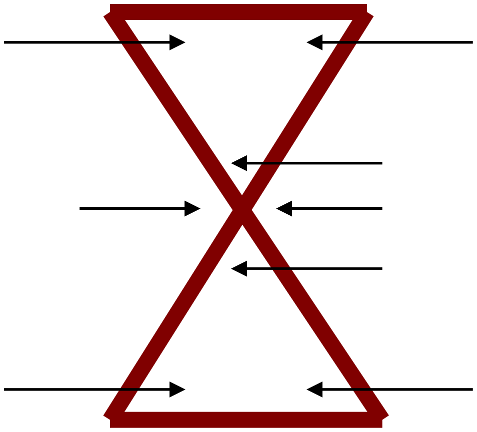
5 angles



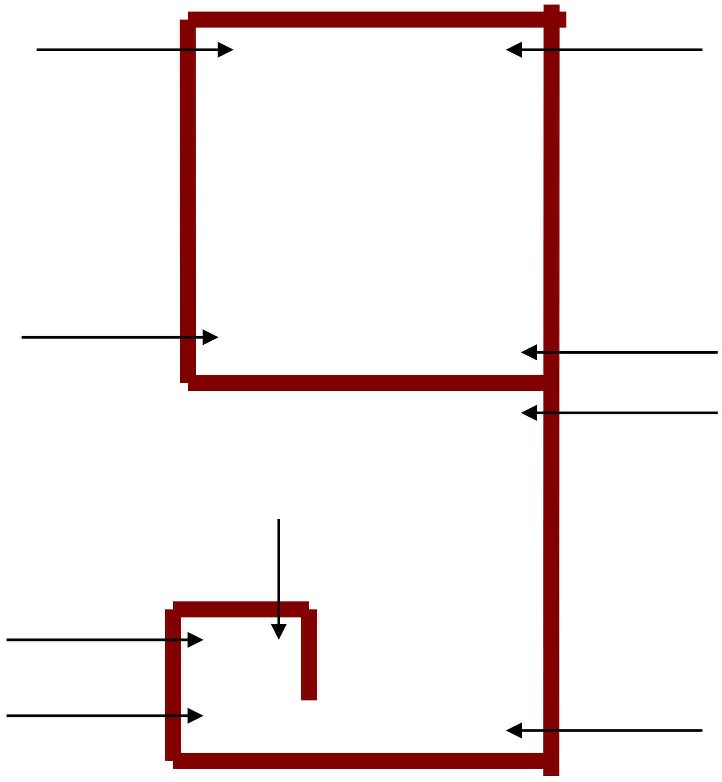
6 angles



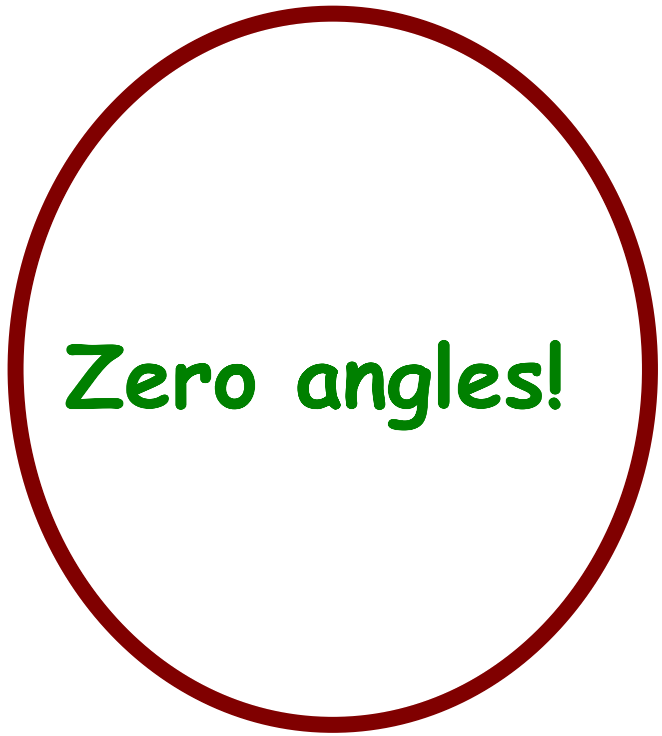
7 angles



8 angles



9 angles


Zero angles!

Moral of the story of 0,1,2…9

- *Merely symbols* chosen to express the concept of a number, though **there is reason** to the symbols.
- Mathematical symbols have changed over the years, and they may change again in the future. The symbols we use depend on our need to determine the value of the quantities with which we work.
- Development of numerals somewhat parallels the development of language as a necessity to quantify the value of quantities which we rely on
  - With more advanced education, our need to expand quantification skills advances, thus stressing the importance of mathematics

Discussion – What is the difference?

- Procedural knowledge
- Conceptual knowledge
  - *Example* – quadratic equation

2.2 Place Value

- Compare the following numbers
  - 4.7
  - 4.70
  - Megan and Donna video
- (Open video)
  - Conceptual versus procedural knowledge

The Place Value System

- The place value system
  - In a **place-value system**, the value of a digit in a numeral is determined by its position in the numeral
  - In our base ten system, the whole-number place values result from groups of ten—ten ones, ten tens, ten hundreds, and so on. This is because our system works fine until we have ten of something because there is no single digit meaning ten. So, for example, when we reach ten ones, that's when we name one ten, with zero ones left over.

PVS - Example

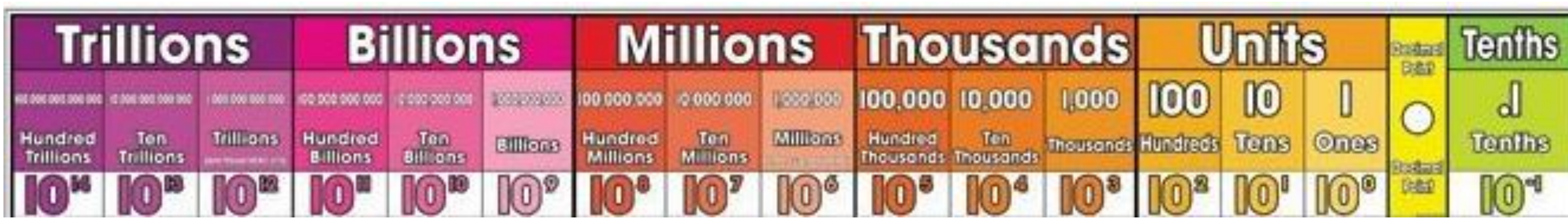
In 506.7, the 5 is in the hundreds place, so it represents five hundred. The 0 in 506.7 is in the tens place, so it represents 0 tens or just zero. The 6 is in the ones place, so it represents 6 ones or six. And the 7 is in the tenths place, so it represents seven-tenths. The complete 506.7 symbol then represents the sum of those values: five hundred six and seven-tenths.

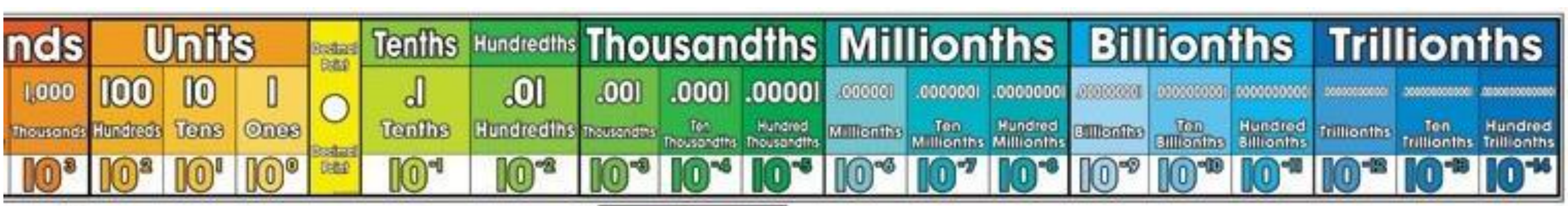
Zero as a placeholder

- Do the zeros hold any meaning in the following numbers?

- 000327
  - 327
- 62.100000
  - 62.1
- 420000
  - 420
  - 42
- 10006
  - 16

Place Value System





Class Activity

- Place value handout

- *Example*: See the difference
  - How many ten dollar bills does the 4 in $243 represent?
    - Four ten dollar bills
  - How many ten dollars bills are in $243?
    - Four is not appropriate with respect to this question
    - Either twenty four or 24.3 *depending on the context*.

2.3 – Bases other than 10

Think About . . .

We use a base-ten system of counting because we have ten fingers. Other cultures have used other bases. For example, some Eskimos were found to count using base five. Why would that be? What other bases might have been used for counting?

- *Recall –*
  - 12 = 30 in "base 4"
- How do alternate base systems work?

Activity

- Andrew’s apple farm

Activity

- Andrew’s apple books

2.3 Bases Other Than Ten

Think About . . .

We use a base-ten system of counting because we have ten fingers. Other cultures have used other bases. For example, some Eskimos were found to count using base five. Why would that be? What other bases might have been used for counting?

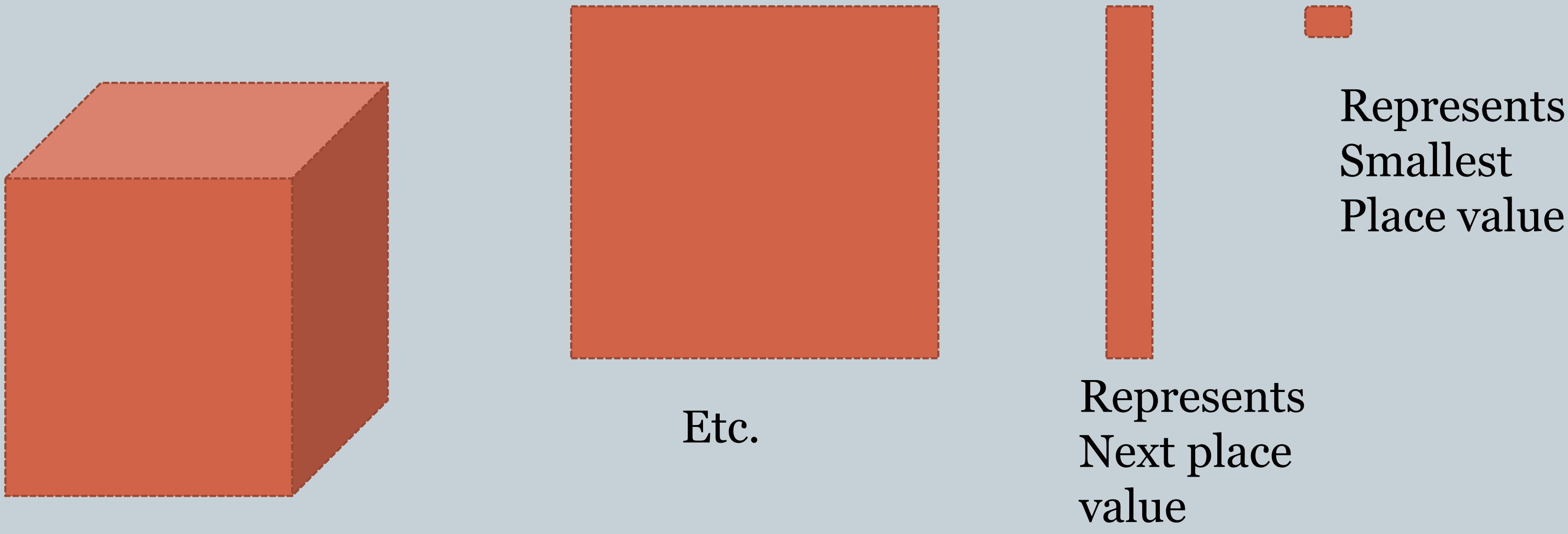
- Different bases are different ways of counting (amount of counting numbers is equal to the base number). They also provide insight into the place value system and how it works in other bases than ten.
- *Note*: In a base, there is no digit to express that value. For instance, in our normal system, base 10, there is no symbol for ten. So, in base six for example, there would be no "6"

Counting in bases

- How to count in base six
  - 0, 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21 … 50, 51, 52, 53, 54, 55, 100, 101, 102, …
- Base eight?
- Base two?

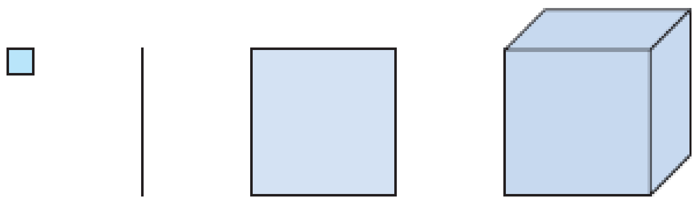
Representation = “Base Block Diagrams”

- We call them
  - Right to left “small”, “straight/long”, “flat”, “cube”
- *Examples* (using doc cam and blocks)
  - Base 10
  - Base 4



Base Block Diagrams

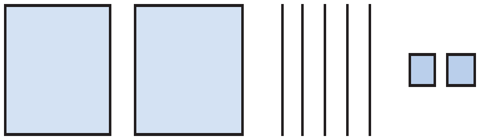
It is helpful to think of other bases using base blocks. We represent them as follows:



These are labeled as “small cube,” “long,” “flat,” and “large cube,” respectively.

Base four, for example, would have zero, one, two or three of any of the given types of blocks.

Discussion

1. Here is a representation of a number: 

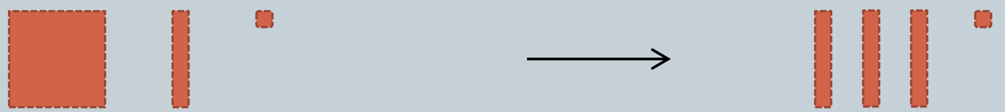
   Which bases could use this representation if it is in the final form, with no more "trades" possible? Why? What are some possible numbers that can be represented by this drawing?

2. In base eight, how many small cubes are in a long? How many in a flat? How many in a large cube? How many longs in a flat? How many flats in a large cube? Answer the same questions for base ten; for base two.

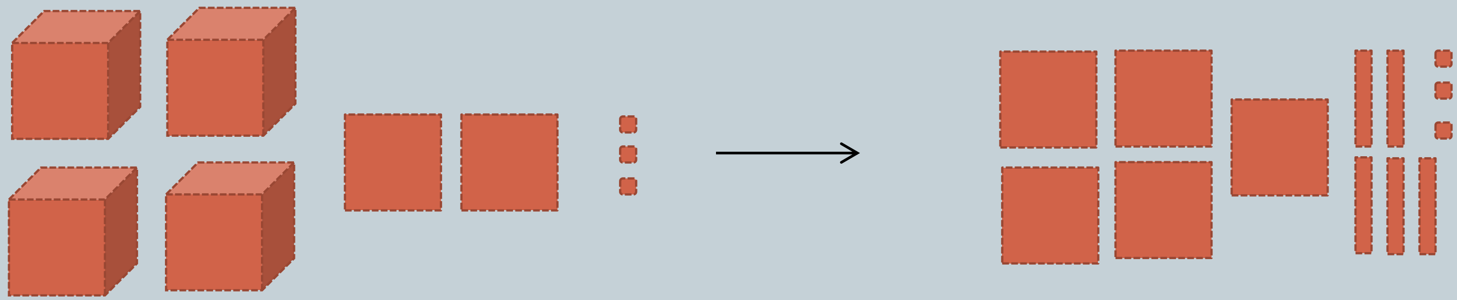
Think about converting between

- If you get confused, try to think about it like Andrew's Apple Farm activity
- The following diagrams are in the respective base. Think about how you would convert between bases.

Five | Ten



111_{five} → 31

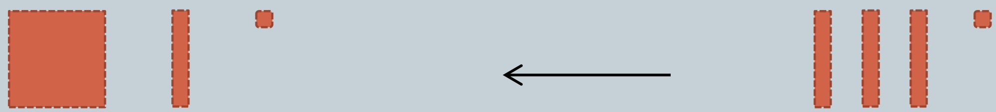


4203_{five} → 553

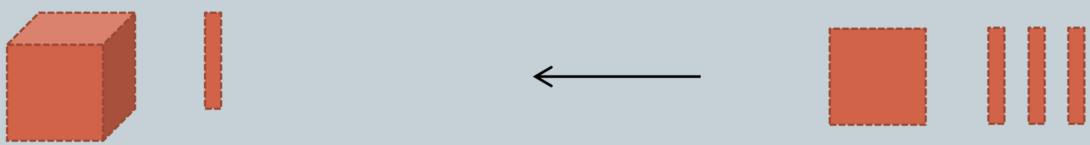
Think about converting between

- If you get confused, try to think about it like Andrew's Apple Farm activity
- The following diagrams are in the respective base. Think about how you would convert between bases.

Five | Ten



111_{five} | 31



1010_{five} | 130

Group Practice!

- Bases practice sheet
  - Diagrams
  - Converting into ten
  - Converting from ten

Formulaic Conversion? Examples

6 to 10

$$3215_{six} = 3^*(6^3) + 2^*(6^2) + 1^*(6^1) + 5^*(6^0)$$
$$= 3^*216 + 2^*36 + 1^*6 + 5^*1$$
$$= 731_{ten} = 731$$

10 to 6

- $3215_{ten} = ?_{six}$

Important: Understand place values in base six

3215_{ten} = __ __ __ __ __ __ __ .

6^6 6^5 6^4 6^3 6^2 6^1 6^0

$6^5 = 7776$ $6^4 = 1296$ $6^3 = 216$ $6^2 = 36$

10 to 6

$3215_{ten} = ?_{six}$

3215_{ten} = __ __ __ __ __ __ __ .

6^6 6^5 6^4 6^3 6^2 6^1 1

$6^5 = 7776$ $6^4 = 1296$ $6^3 = 216$ $6^2 = 36$ $6^1 = 6$

Figure out place vals from biggest to smallest (note all operations are in base 10)

$2 * 6^4 = 2592$ → $3215 - 2592 = 623$

$2 * 6^3 = 432$ → $623 - 432 = 191$

$5 * 6^2 = 180$ → $191 - 180 = 11$

$1 * 6^1 = 6$ → $11 - 6 = 5$

$5 * 1$

3215_{ten} = 0 0 2 2 5 1 5 = 22515_{six}

6^6 6^5 6^4 6^3 6^2 6^1 1 .

Notes!

- Converting from one base **into ten** is easier because we understand base ten really well.
  - Procedural way to convert is fairly simple

- Converting from base ten **into a different base** is trickier. It will help to make a place-value chart like ($6^5 = 7776$ $6^4 = 1296$ $6^3 = 216$ $6^2 = 36$ $6^1 = 6$)
  - Don't forget every base has a ones place. Every base system has "1" (the ones place) as its focal point.
  - Don't forget place holder zeros
  - Procedural way to convert is more complicated, understanding is critical for this process

What about the Decimal Point? Fractions?

- What does the following represent? Could we convert it into base 10?

$$40.13_{five}$$

Why are we studying other bases?

Understanding place value is considered to be foundational to elementary school mathematics. Yet place-value instruction in schools is often superficial and limited to studying only the placement of digits. Thus, children are taught that the 7 in 7200 is in the thousands place, the 2 is in the hundreds place, a 0 is in the tens place, and a 0 is in the ones place. But when asked how many hundred dollar bills could be obtained from a bank account with $7200 in it, or how many boxes of ten golf balls could be packed from a container with 7200 balls, children almost always do long division, dividing by 100 or by 10. They do not read the number as 7200 ones, 720 tens, or 72 hundreds, and certainly not as 7.2 thousands. But why not? These are all names for the same number, and the ability to rename in this way provides a great deal of flexibility and insight when working with the number.

One activity-centered primary program incorporates many activities involving grouping by twos, by threes, etc., even before extensive work with base-ten groupings, to accustom the children to counting not just one object at a time, but groups each made up of several objects. Ungrouping needs to be included also. That is, 132 could be regarded as 1 one hundred, 3 tens, and 2 ones. Or, it could be regarded as 1 one hundred and 32 ones. Here, the 3 tens are "unbundled" to make 30 ones. Regarding a group made up of several objects as one thing is a major step that needs instructional attention.

2.4 – Operations on Different Bases

- Addition → use block diagrams
  - Lingo … "combining blocks"
- Subtraction → use block diagrams
  - Lingo … "take away blocks"

- Do these examples in class
  - Act out 112 + 101 in base three using blocks
  - Then draw 112 + 101 in base three on board

Practice +/- in alternate bases