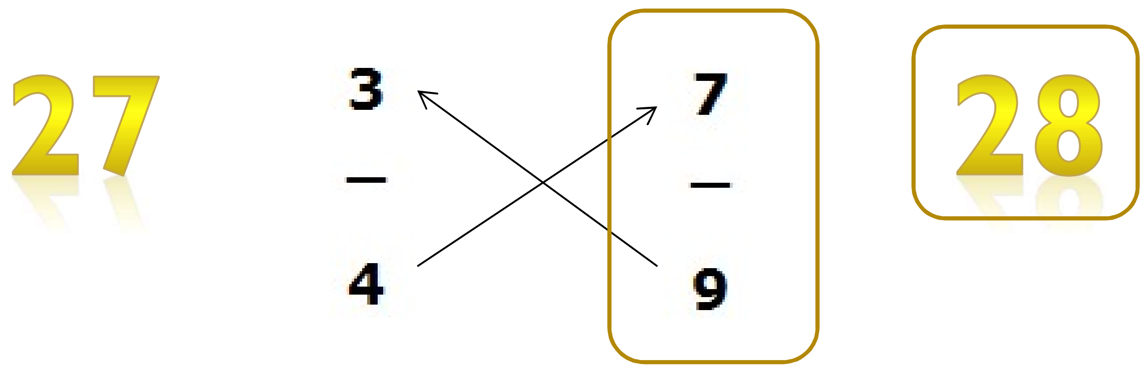
Chapter 8

MULTIPLICATIVE REASONING

Fraction Comparison & Cross Multiply Rule

- Which is bigger (use cross multiply rule)



- Why does this rule work???
  - Comparing using the LCD

Multiplicative reasoning (versus additive)

- Zeus planted two trees last year. The first was originally 2 feet tall and the second was originally 3 feet tall. Zeus went back to check on them now, and the first is now 4 feet tall and the second is now 5 feet tall. Which grew more?

- The question is actually a bit ambiguous
  - If we look at the difference, each tree grew 2 feet.
  - The other way to consider the situation is that the first tree doubled in height and the second tree did not.
    - *Note* – usually when this question is asked it is implied that we do an additive comparison…this happens all the time without noticing

Multiplicative comparison

Given two quantities, whenever we want to determine *how many times as large one of them is than the other one,* we do a **multiplicative comparison** of the two quantities.

EXAMPLE 1

If length A has a value of 12 meters and length B has a value of 3 meters, the *multiplicative comparison* of A to B is 4. The multiplicative comparison of B to A is $\frac{1}{4}$ because length B is $\frac{1}{4}$ of length A. An *additive comparison* of the two lengths might say that A is 9 meters longer than B.

Given two quantities, we can compare them either additively or multiplicatively.

EXAMPLE 2

If a city is expected to grow from 100,000 people to 130,000 people, we could claim that either the population will grow by 30,000 people (an additive comparison) or the population will become 130% of its current value (from the multiplicative comparison of the expected 130,000 persons with the 100,000 current population).

Discussion

- Consider a population growth of 1,000
  - What does that look like?
- If our town's population was originally 1,000, what is the growth of the population?
- How about if our town's population was originally 10,000? What about 100,000? What about 100?
  - We have 1,000 going to 2,000 (200% of original (100% growth))
  - And 10,000 going to 11,000 (110% of original (10% growth))
  - And 100,000 going to 101,000 (101% of original (1% growth))
  - And 100 going to 1,100 (1100% of original (1000% growth))

Multiplicative Comparison Group Activity

ACTIVITY 2 As Time Goes By

▶ Today is Sally's birthday. She is 7 years old. At some time in the future, John will have his 39th birthday. At that time, he will be 3 times as old as Sally. How old is John now?[2] ◀

a. Identify all the quantities in the situation, including those whose values are not known.

b. What does the 3 in the problem refer to? What quantities are being compared?

c. How would you as a teacher respond to a student who says that since John is 3 times as old as Sally, and Sally is 7 years old, John must be 21 years old now?

d. How much time will elapse between now and the time when John is 39 years old?

e. How old is John now?

f. What is the difference between Sally's and John's ages when John is 39? Twenty-five years from now? Now?

g. What can you conclude about the difference between John's and Sally's ages as time goes by?

h. As long as John is alive, will he always be 3 times as old as Sally? Explain.

Ratio

The result of comparing two quantities multiplicatively is called a **ratio**. If x is the value of quantity A and y is the value of another quantity B, then the ratio $x{:}y$, or $\frac{x}{y}$, tells us how many times as large A is as B. The ratio $x{:}y$ is often pronounced "x to y" or "x is to y."

The context usually makes clear whether to use the $x{:}y$ or $\frac{x}{y}$ notation.

- With this in mind, look at example 2. What was the ratio in fractional form? In colon form?

EXAMPLE 2

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  - The multiplicative comparison was 130%

Ratio

•Do ratios refer to multiplication, or division?

•Trick question…they can be viewed in both lights

EXAMPLE 4

If time period A is 192 hours and time period B is 12 hours, then A is 16 times as long as B. The 16 comes from $\frac{192}{12}$, or its link, $192 \div 12$. For this reason, an elementary school book might say that a ratio is a comparison of two amounts by division.

•12*? = 192
•192/12 = ?

Activity

- Candy bar and ratios problem

Chapter 9

RATIOS, RATES, PROPORTIONS, AND PERCENTS

9.1 Ratio as a Measure

- In chapter 8 we use ratios to compare quantities
- In this chapter it we will develop that idea in new contexts
- Consider the following question
  - "At some time before the end of the day, there remains 4/5 of what has elapsed since the day began at midnight."
    - 1. Is it before noon, or after noon? Why?
    - 2. The number of hours that have elapsed since the beginning of the day is ____ times the number of hours that remain.
    - 3. What is the time described?
- What does the original statement say?
  - "4/5 of the time that has elapsed remains"
  - This means some amount of time has elapsed, and 4/5 of that amount remains
    - See next slide

Example question involving ratios

- "At some time before the end of the day, there remains 4/5 of what has elapsed since the day began at midnight."
  - 1. Is it before noon, or after noon? Why?
  - 2. The number of hours that have elapsed since the beginning of the day is ____ times the number of hours the remain.
  - 3. What is the time described?

- "4/5 of the time that has elapsed remains"
  - This means some amount of time has elapsed, and 4/5 of that amount remains
  - What if we added those two quantities together? We would get total time.
    - Total time = $t + \frac{4}{5}t$ where t stands for "time elapsed from the beginning of the day until now"
- It should be fairly clear now that we are past noon since more time has elapsed since the beginning since $1 > \frac{4}{5}$ … the ratio should be seen as 4:5 and 5:4
- The number of hours that have elapsed since the beginning of the day is 5/4 times the number of hours the remain. (the inverse ratio)
- What is the time described?
  - Since the ratio is 5:4, we have 5 out of 9 total pieces. $\frac{5}{9} * 24\ hours = 13\ \frac{1}{3}$ hours so 1:20pm

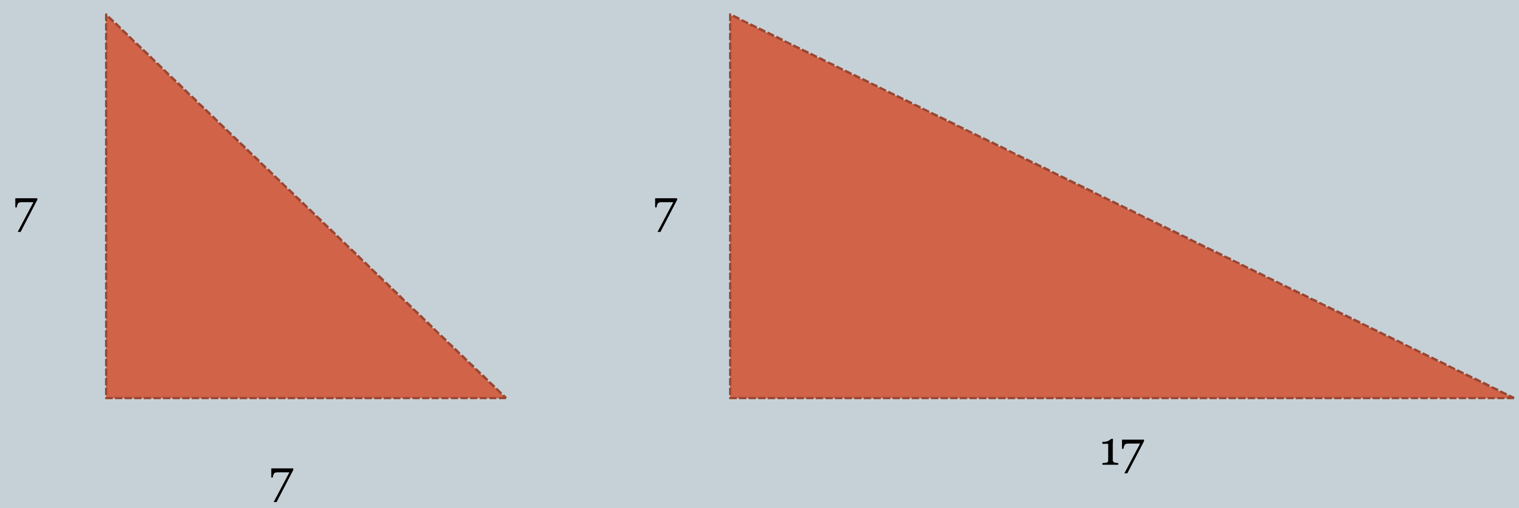
Discussion

- Consider the following problem: “A new housing subdivision offers rectangular lots of three different sizes. Which of the lots is **most square**?”
- In other words, from above, which would appear to be the closest to a square?
  - A. 75 feet by 114 feet
  - B. 455 feet by 508 feet
  - C. 185 feet by 245 feet

  - Stuck? Try considering the ratios.
  - Think about … what is the ratio of a square?

Ratios = Multiplicative comparisons

- Ratios can be used to compare sizes
- Sides of a square to compare "squareness"
- Dimensions of a slope to compare "steepness"



Comparing Ratios

ACTIVITY 1 Orange Juicy

The orange drink in Pitcher A is made by mixing 1 can of orange concentrate with 3 cans of water. The mixture in Pitcher B is made by mixing 2 cans of orange concentrate with 6 cans of water. Which will taste more "orangey": the mixture in Pitcher A or the mixture in Pitcher B?[4]

- (A) Which is "more orangey"? Explain.

Comparing Ratios

- *Remember* - Ratio comparisons tie directly to fraction comparisons!

ACTIVITY 1 Orange Juicy

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- Ratios… 1:3 and 2:6 (or 3:1 and 6:2)
- Corresponding fractions
  - 1/3 and 2/6 …. these are equivalent
  - 3/1 and 6/2 … these are also equivalent
- Both comparisons are equivalent fractions → equivalent ratios!

Discussion – Comparing Ratios

b. How would you, as a teacher, respond to a student who says that the mixture in Pitcher A is more orangey because less water went into making it?

c. In a fifth-grade class some students reasoned like the student in part (b). Other students in the same class argued that the mixture in Pitcher B is more orangey because it has more orange concentrate. How would you settle the argument? What is wrong with the reasoning of the students in each group?

d. There was another student who argued as follows:

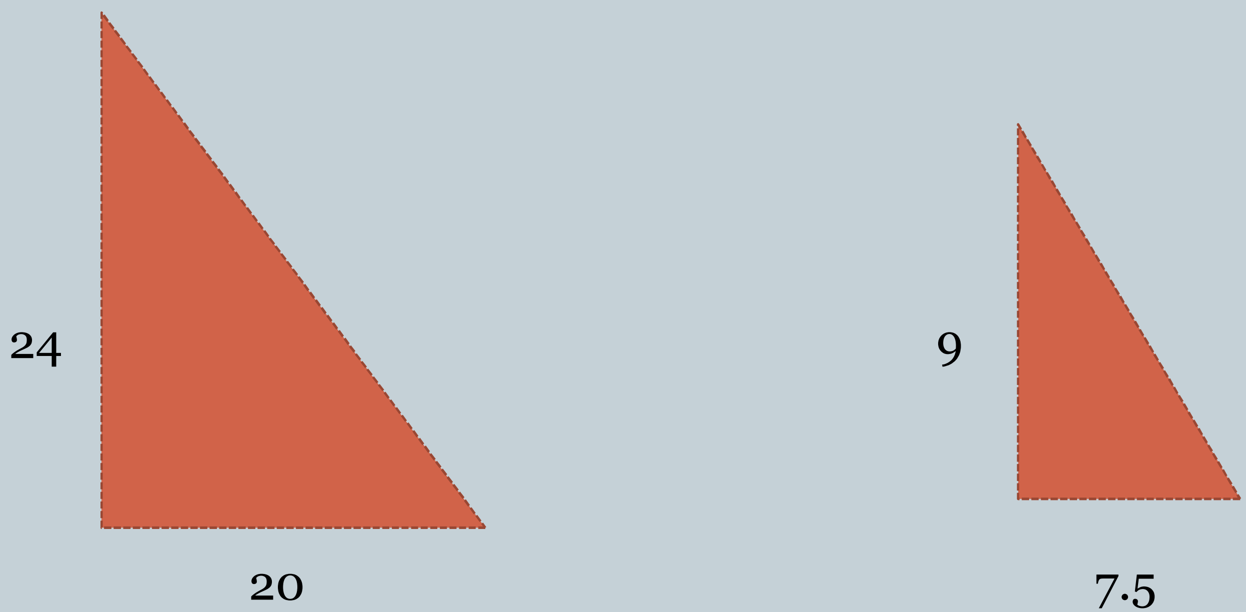
> Pour 1 can of orange concentrate and 1 can of water into Pitcher A. Take 2 cans of orange concentrate and 2 cans of water and pour them into Pitcher B. The two mixtures are equally orangey because they are made with equal parts orange and water, that is, 50-50. Now there are only 2 cans of water left to go into Pitcher A and 4 cans of water left to go into Pitcher B. Because the mixtures are the same strength, when you add 4 cans of water to the mixture in Pitcher B, it will be more watery than the mixture in Pitcher A, which gets only 2 more cans of water. Therefore, the mixture in Pitcher A is more orangey.

How would you deal with this student's thinking?

e. How can "oranginess" be quantified in this situation to facilitate the comparison?

Ratios = Multiplicative comparisons

- Ratios can be used to compare sizes
- Dimensions of a slope to compare "steepness"
  - Which of the following slopes is "steeper?"



Proportional / Unit Ratio

- A proportion is a statement that two ratios are equal to one another. The quantities are said to be "**proportional**"
  - The two ratios in the orange juice problem were equivalent, i.e. proportional 1:3 as 2:6
- A "**unit ratio**" is a ratio where <u>the second value is one</u>
  - 4:1
  - 100:1
- You can write any ratio as a unit ratio (or unit fraction)

$$1:3 \textit{ is the same as } \frac{1}{3}:1, \textit{ or } \frac{\frac{1}{3}}{1}.$$

Practice

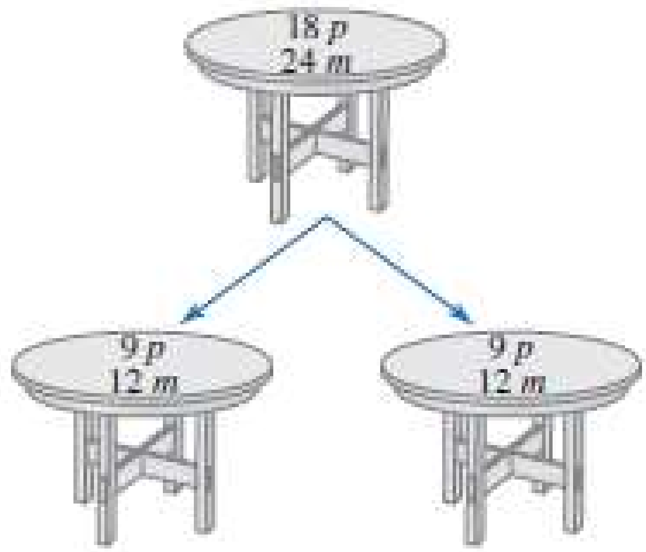
- How would I write the following as unit ratios?
  - 5:1 →
  - 8:2 →
    - 4:1 or "four to one"
  - 100:10 →
    - 10:1
  - 59:49 →
    - $\frac{59}{49}:1$
  - 1:4 →
    - $\frac{1}{4}:1$
  - 2:5 →
    - $\frac{2}{5}:1$
  - 30:100 →
    - $\frac{3}{10}:1$

Proportional Reasoning

- You can use proportions to solve a problem that involves multiplicative comparisons and groups of different sizes

ACTIVITY 3 Pepperoni to Go![Y]

Luigi's is a pizza parlor that caters to the local college crowd. The 24 members of the chess club come in to celebrate. Eighteen pizzas had been ordered in advance (all the same size). None of the tables will hold 24 persons, so they sit at two tables, each with 9 pizzas and 12 members.



Discussion – When NOT to use Proportions

- Explain why you **shouldn't** use proportional reasoning on the following problems.
  - 1. Blazer drove 72 miles during the first hour of his trip. How long will it take him to drive the entire 720 miles of his trip?
  - 2. In a pie eating contest, McLovin ate two pies in the first 5 minutes. How many pies can he eat in 2 hours?
  - 3. It took Nacho 20 minutes to complete the first 10 problems of his 30 problem exam. How long will it take him to finish?
  - 4. Dr. Phil can mow the lawn in 45 minutes. If Mrs. Phil helps him, how long will it take them together?

Rate

- A "**rate**" is a ratio of quantities that change without changing the value of the ratios
- Let fix the previous slides problems.
  - 1. Blazer drives 36 miles per hour (on average). How long will it take him to drive the entire 720 miles of his trip?
    - 36 miles/hour → 360 miles/10 hours → 720 miles/20 hours
  - 2. In a pie eating contest, McLovin can eat two pies every twenty minutes. How many pies can he eat in 2 hours?
    - 2 pies/20 mins → 6 pies/hour → 12 pies/2 hours
  - 3. In10 minutes Nacho solves 6 problems (on average). If he is given a 30 problem exam, how long will it take him to finish?
    - 6 problems/10 mins → 30 problems/50 mins
  - 4. Dr. Phil can mow the lawn in 45 minutes. Mrs. Phil can mow the lawn in 30 minutes. If she helps him, how long will it take them together?

Ratio concept you should remember

Dr. Phil can mow the lawn in 45 minutes. Mrs. Phil can mow the lawn in 30 minutes. If she helps him, how long will it take them together?

- Lawn:DrPhil → 1:45
- Lawn:MrsPhil → 1:30
  - $\frac{1}{45} + \frac{1}{30} = \frac{2}{90} + \frac{3}{90} = \frac{5}{90} = \frac{1}{18}$
  - So Lawn: (D+M) = 1:18 → it will take them 18 minutes
- Why doesn't this work the other way?
  - D:Lawn → 45:1 (45 minutes in 1 lawn? Doesn't make sense)
  - M:Lawn → 30:1
  - 45+30 = 75 → 75:1 … this answer is longer, so it doesn't make sense
- To solve, <u>we need to add their rates</u> together

Percents and Ratios

- A "**percent**" is a ratio for which the value of the second quantity is understood to be 100
  - This is an alternate definition for percent
  - Examples
    - 4:100 → 4/100 → 0.04 → 4%
    - 10:100 → 10/100 → 1/10 → 0.1 → 10%
    - 77:100 → 77/100 → 0.77 → 77%

Comparing Numbers using Percentages

- Suppose you got 15.5 out of 20 on your first quiz. Suppose you got 59 out of 75 on your second quiz. How do you figure out which quiz you did better on?
  - You can figure out the percent of the problems that you got correct
  - $15.5:20 \rightarrow \frac{15.5}{20} \rightarrow \frac{15.5*5}{20*5} \rightarrow \frac{77.5}{100} \rightarrow 77.5:100 \rightarrow 77.5\%$
  - $59:75 \rightarrow \frac{59}{75} \rightarrow \frac{59*\frac{100}{75}}{75*\frac{100}{75}} \rightarrow \frac{59*\frac{4}{3}}{100} \rightarrow 78.\bar{6}:100 \rightarrow 78.\bar{6}\%$
  - So you "did better" on the second quiz since you got a higher percentage of the questions correct.

Another percentage question

- The boss says, “You remember when business was bad last year, I had to cut everyone’s pay by 10%? Well, business is better, so I can raise your pay by 10% now. That will put you back to where you were before the cut.”
  - Is the boss correct?

Review

- A **ratio** A:B is simply a <u>multiplicative comparison</u>
  - We can simplify a ratio as we would a fraction
- We can write any ratio as a **unit ratio** C:1
- A **rate** is a ratio that involves units
  - Ex: 130 miles in 2 hours
- A **unit rate** is a rate compared to 1.
  - Ex: Simplifying the previous rate, 130:2 → 65:1 → 65 mph
- A **percentage** is a ratio to 100.
  - Percentages that we encounter, just like ratios, are often multiplicative comparisons of one quantity to another. Often the multiplicative comparison is signified by "of."

Solving Percentage Problems

- Since a problem involving a percentage must involve a multiplicative comparison (what is the % referencing?), they can always be solved using:

$$\frac{is}{of} = \frac{\%}{100}$$

- Why? The comparison of the two numbers is proportional to the indicated percentage.

Issues for Learning Proportions

- Most adults use a **cross-multiplication** method to solve for a missing value in a proportion.
- However, research shows that even when this strategy is explicitly taught to 11- and 12-year-olds, they often become confused and are unsuccessful w
- If left on their own to solve such used the *unit method*, such as w
  - John purchased 24 loaves of bread at a total price of $26. If he wishes t much will he have to spend?
- The following answer is typical:

What is the unit method??

1. Find the given ratio/rate.
2. Convert it to a unit ratio/rate
3. Multiply or divide the remaining quantity by the unit ratio/rate depending on the given comparison.

► If John is paid $26.00 for 24 loaves, this means that he must have been paying a little over $1.00 for every loaf. The exact amount is 26 divided by 24, or $1.08\overline{3}$. If John wants to buy 30 loaves next week, that means he will have to pay 30 times $1.08\overline{3}$, or $32.50. ◄

% Increase and Decrease?

- One confusing element of a % discount or % increase is that it is an additive comparison in % upon a multiplicative comparison
- It helps me to remember that % are always referent to 100%
  - "15% off" means "15% off of 100%" or "85%" which is the multiplicative comparison
  - "50% increase" means "50% added to 100%" or "150%" which is the multiplicative comparison

Solving Percentage Problems Multiplicatively

Activity – For each problem, consider what single percentage describes the multiplicative comparison.

1. Say you scored 63 out of 75 points on a history test. What percentage of the test did you get correct?
2. A pair of pants is on sale for 30% off. If its original price was $49.99, what is its sale price?
3. At work you started with a salary of $50,000. Last year, you got a 5% raise. Then this year, you got another 10% raise. What is your salary today? Would it have been the same as a single 15% raise?
4. A newspaper says that the population of squirrels in a local park has increased by 150% over the past year. If there were 600 squirrels in the park last year, how many are there now?

Ratios Can Be “Annoying”

The following problem[5] is from an SAT exam, and very few students solved it. Can you solve this problem?

A flock of geese on a pond were being observed continuously.

At 1:00 PM, $\frac{1}{5}$ of the geese flew away.

At 2:00 PM, $\frac{1}{8}$ of the geese that remained flew away.

At 3:00 PM, 3 times as many geese as had flown away at 1:00 PM flew away, leaving 28 geese on the pond. At no other time did any geese arrive, fly away, or die. How many geese were in the original flock? ◄

What Should We Try To Develop?

Susan Lamon, who has been investigating proportional thinking for many years, has listed the following characteristics of proportional thinkers. Use them to evaluate whether or not you are a proportional thinker.

1. They can think both in terms of unit rates, such as 25 miles per gallon of gas, and in terms of multiple units, such as $5.15 per 6 bottles of water.
2. They are more efficient problem solvers.
   - If 6 bottles of water cost $5.15, and they wanted to know the cost of 24 bottles of water, they would think: 4 groups of 6 is 4 3 $5.15. Finding the unit price per bottle and then multiplying by 24 is a previously learned method of finding the price of 24 bottles, but it is less efficient and more prone to errors.
3. They can use partitioning to help them.
   - For example, if 3 people share 5 pizzas, then each person gets 5/3 pizzas. That is, each person gets 1/3 of the total 5 pizzas, which is 1/3*5=5/3 pizzas.
4. They can think flexibly about quantities and find unit quantities.
5. They are not afraid of fractions and decimals and can think flexibly with these numbers.
6. They can mentally compute with fractions, decimals, and percents.
7. They can identify everyday situations where proportions are not useful.
8. They can solve both missing-value problems and comparison problems by reasoning about them, not just rotely using the cross-multiplication strategy.