Practice Exam #3 Key

As usual, be cautious for errors. I typed this up relatively quickly! Let me know if you suspect something is off.

[1A] Apply ratio test (since $n!$ is present).

$$\lim_{n\to\infty}\left|\frac{n+1}{4^n\cdot 4}\cdot\frac{4^n}{n}\right| = \frac{1}{4}\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \frac{1}{4} < 1$$

Therefore the series $\sum \frac{n}{4^n}$ converges absolutely by the ratio test.

[1B] Apply comparison test or limit comparison test (since series resembles p-series).

$$b_n = \frac{3n}{\sqrt{4n^3}} = \frac{3}{2\sqrt{n}},\quad \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{3n+1}{\sqrt{4n^3+2n^2-3}}\cdot\frac{2\sqrt{n}}{3}\right) = \lim_{n\to\infty}\left(\frac{6n^{3/2}+2n^{1/2}}{\sqrt{36n^3+18n^2-27}}\right) = 1$$

Therefore the series $\sum \frac{3n+1}{\sqrt{4n^3+2n^3-3}}$ diverges by Limit Comparison Test since $\sum b_n$ is a divergent p-series.

[2] This series is geometric.

$$\sum_{n=1}^{\infty} 7\left(-\frac{1}{3}\right)^n = -\frac{7}{3}+\frac{7}{9}-\frac{7}{27}+\frac{7}{81}-\cdots$$

Since $|r| = \frac{1}{3} < 1$ this series converges and it converges to $\frac{\frac{7}{3}}{1-\left(-\frac{1}{3}\right)} = \frac{\frac{7}{3}}{\frac{4}{3}} = \frac{7}{4}$

[3] Apply the alternating series test (since $(-1)^n$ is present). $b_n = \frac{1}{n^3+1}$

(i) $\frac{1}{n^3+3n^2+3n+1} \le \frac{1}{n^3+1} \Rightarrow n^3+1 \le n^3+3n^2+3n+1 \Rightarrow 0 \le 3n^2+3n$ True $\forall\, n \ge 1$ ■

(ii) $\lim_{n\to\infty}\left(\frac{1}{n^3+1}\right) = 0$ ■

Therefore $\sum \frac{(-1)^n}{n^3+1}$ converges by the Alternating Series Test.

Absolutely? Yes because $\frac{1}{n^3+1} \ge \frac{1}{n^3}$. Therefore $\sum \frac{1}{n^3+1}$ converges by comparison to p-series.

Therefore $\sum \frac{(-1)^n}{n^3+1}$ converges absolutely. (The alternating series test wasn't necessary in this case)

[4A] Apply the alternating series test (since $(-1)^n$ is present). $b_n = \frac{n}{n^2+1}$

(i) $\frac{n+1}{n^2+2n+2} \le \frac{n}{n^2+1} \Rightarrow n^3+n^2+n+1 \le n^3+3n^2+2n \Rightarrow 1 \le 2n^2+n$ True $\forall\, n \ge 1$ ■

(ii) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = 0$ ■

Therefore $\sum \frac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test.

Absolutely? No because $\frac{n}{n^2+1} \ge \frac{n}{n^2+n^2} = \frac{1}{2n}$. Therefore $\sum \frac{n}{n^2+1}$ diverges by comparison to p-series.

Therefore $\sum \frac{(-1)^n n}{n^2+1}$ converges conditionally.

[4B] Apply comparison test or integral test. $u = \ln x\,, du = \frac{1}{x}dx$

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_0^{\ln t} u\,du = \lim_{t\to\infty}\left[\frac{u^2}{2}\right]_0^{\ln t} = \lim_{t\to\infty}\left[\frac{(\ln t)^2}{2}\right] = \infty \rightarrow \text{Diverges}$$

Therefore $\sum \frac{\ln n}{n}$ diverges by the integral test.

[5A] Apply the ratio test.

$$\lim_{n\to\infty}\left|\frac{x^n x}{(n+1)5^n 5}\cdot\frac{n5^n}{x^n}\right| = \frac{|x|}{5}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{|x|}{5} < 1 \quad \Longrightarrow \quad |x| < 5 \quad \Rightarrow (-5,5)$$

Ratio test is inconclusive when $L = 1$ or $x = \pm 5$.

When $x = 5$ → $\sum \frac{1}{n}$ diverges. When $x = -5$→ $\sum \frac{(-1)^n}{n}$ converges.

Therefore the interval of converges is $[-5,5)$ and $R = 5$.

[5B] Apply the ratio test.

$$\lim_{n\to\infty}\left|\frac{3^n 3 x^n x}{(n+1)^2}\cdot\frac{n^2}{3^n x^n}\right| = 3|x|\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = 3|x| < 1 \quad \Longrightarrow \quad |x| < \frac{1}{3} \quad \Rightarrow \left(-\frac{1}{3},\frac{1}{3}\right)$$

Ratio test is inconclusive when $L = 1$ or $x = \pm\frac{1}{3}$.

When $x = \frac{1}{3}$ → $\sum \frac{1}{n^2}$ converges. When $x = -\frac{1}{3}$→ $\sum \frac{(-1)^n}{n^2}$ converges.

Therefore the interval of converges is $\left[-\frac{1}{3},\frac{1}{3}\right]$ and $R = \frac{1}{3}$.

[6A]

$$\sqrt{e} = e^{\frac{1}{2}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{2!\,2^2} + \frac{1}{3!\,3^3} + \frac{1}{4!\,4^4}\ldots \approx 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{6144} \approx 1.65$$

[6B]

$$\cos\frac{\pi}{5} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{5}\right)^{2n} = 1 - \frac{\left(\frac{\pi}{5}\right)^2}{2!} + \frac{\left(\frac{\pi}{5}\right)^4}{4!} - \frac{\left(\frac{\pi}{5}\right)^6}{6!} + \cdots \approx 0.809$$

[6C]

$$\sin 10° = \sin\frac{\pi}{18} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{18}\right)^{2n+1} = \left(\frac{\pi}{18}\right) - \frac{\left(\frac{\pi}{18}\right)^3}{3!} + \frac{\left(\frac{\pi}{18}\right)^5}{5!} - \frac{\left(\frac{\pi}{18}\right)^7}{7!} + \cdots \approx 0.174$$

[6D]

$$\ln\frac{3}{2} = \ln\left(1 + \frac{1}{2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{2}\right)^n = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots \approx 0.405$$

[7] Banana

[8A] Apply integral test.

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty}\left[-\frac{1}{3x^3}\right]_1^t = \frac{1}{3} \to \text{ Converges}$$

Therefore the series converges by the integral test.

[8B]

$$S_8 = \sum_{n=1}^{8} \frac{1}{n^4} \approx 1.08$$

[8C]

$$\int_9^{\infty} \frac{1}{x^4}\,dx \le R_8 \le \int_8^{\infty} \frac{1}{x^4}\,dx \;\Rightarrow \left[\frac{1}{3(9)^3}\right] \le R_8 \le \left[\frac{1}{3(8)^3}\right] \;\Rightarrow 0.00046 \le R_8 \le 0.00065$$

Therefore S_8 is accurate to the nearest thousandth.

[9A] Converges since $r = \frac{2}{5}$. Converges to $\frac{2}{1-\frac{2}{5}} = \frac{2}{\frac{3}{5}} = \frac{10}{3}$

[9B] Diverges since $r = \frac{3}{2}$

[10A]

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^4}}{3n+4} =^{H}= \lim_{n\to\infty} \frac{\frac{4}{3}\sqrt[3]{n}}{3} = \infty.$$

The sequence diverges. Therefore the series would also diverge by the divergence test.

[10B]

$$a_n = (-1)^{n+1}\frac{n}{n+1}$$

This sequence diverges because it will alternate towards 1. Therefore the series would also diverge by the divergence test.

[11] Apply integral test.

$$\int_1^{\infty} \frac{1}{2x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4x^2}\right]_1^t = \frac{1}{4} \to \text{ Converges}$$

Therefore the series converges by the integral test.

[11A]

$$S_{10} = \sum_{n=1}^{10} \frac{1}{2n^3} \approx 0.60$$

[11B]

$$\int_{11}^{\infty} \frac{1}{2x^3}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{2x^3}\,dx \;\Rightarrow \left[\frac{1}{4(11)^2}\right] \le R_{10} \le \left[\frac{1}{4(10)^2}\right] \;\Rightarrow 0.00207 \le R_{10} \le 0.0025$$

Therefore S_{10} is accurate to the nearest hundredth.

[12A] Converge by Ratio Test.

[12B] Converge by Root Test.

[12C] Diverge by Limit Comparison Test (use $b_n = \frac{1}{n}$ then L'Hospitals)

[12D] Diverge by Divergence Test.

[13] $0.373737\ldots = 0.37 + 0.0037 + 0.000037 + \cdots = \frac{37}{100}\left(1 + \frac{1}{100} + \frac{1}{10000} + \cdots\right) = \frac{37}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{37}{100}\cdot\frac{100}{99} = \frac{37}{99}$