Exam #3 NAME:__________KEY______________

1. Determine whether the given series are absolutely convergent, conditionally convergent, or divergent. Explain.

a. $$\sum_{n=1}^{\infty}(-1)^n \frac{n+4}{\sqrt{n^3+7}}$$

$$\frac{n+5}{\sqrt{n^3+3n^2+3n+8}} \le \frac{n+4}{\sqrt{n^3+7}} \Rightarrow (n^3+7)(n+5)^2 \le (n+4)^2(n^3+3n^2+3n+8) \Rightarrow$$

$$n^5+10n^4+25n^3+7n^2+70n+175 \le n^5+11n^4+43n^3+80n^2+112n+128 \Rightarrow$$

$$48 \le n^4+18n^3+73n^2+42n \ \blacksquare \quad \text{True } \forall\, n \ge 1$$

$$\lim_{n\to\infty}\frac{n+4}{\sqrt{n^3+7}} = \lim_{n\to\infty}\frac{\sqrt{n^3}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n^3}}\right)}{\sqrt{n^3}\sqrt{1+\frac{7}{n^3}}} = \lim_{n\to\infty}\frac{\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n^3}}\right)}{\sqrt{1+\frac{7}{n^3}}} = \frac{0}{1} = 0$$

This series converges by Alternating Series Test.

Its absolute value does not converge by limit comparison test to $\sum \frac{n}{\sqrt{n^3}} = \sum \frac{1}{n^{1/2}}$ which is a divergent p-series.

$$\lim_{n\to\infty}\frac{n+4}{\sqrt{n^3+7}} \cdot \frac{n^{1/2}}{1} = 1$$

Therefore this series is **conditionally convergent**.

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b. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

This series is **absolutely convergent** because the absolute value series is $\sum \frac{1}{n^2}$ a convergent p-series.

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2. Determine if the following sequence converges or diverges. If instead it were a series, would anything change about its convergence or divergence? Explain.

$\{e^{-n}\}_{n=0}^{\infty}$

$$\lim_{n\to\infty}\frac{1}{e^n} = 0$$

The sequence **converges** to zero.

$$\sum_{n=0}^{\infty}\frac{1}{e^n} = \sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n$$

Since the series is geometric with $r=\frac{1}{e}$, thus the series also **converges** to $\frac{1}{1-\frac{1}{e}} = \frac{e}{e-1} \approx 1.58$

3. Find the radius and interval of convergence of the given power series.

$$\sum_{n=1}^{\infty} \frac{n}{n!} x^{2n}$$

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{2n+2}}{(n+1)!} \cdot \frac{n!}{nx^{2n}} \right| = \lim_{n\to\infty} \left| \frac{n!\,(n+1)x^{2n}x^2}{(n+1)n!\,nx^{2n}} \right| = |x^2| \cdot \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0 < 1$$

This power series **converges always** for $x \in \mathbb{R}$ i.e. $(-\infty, \infty)$. Its radius is $R = \infty$.

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4. Use the integral test to show that the following series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

$$\int_1^{\infty} \frac{1}{x^5} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^5} dx = \lim_{t\to\infty} \left[-\frac{1}{4x^4}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{4t^4} + \frac{1}{4}\right] = \frac{1}{4}$$

So this series **converges** by the integral test.

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a. Estimate the sum of the series (S) by calculating the 8th partial sum (S_8) with your calculator.

$$\sum_{n=1}^{8} \frac{1}{n^5} = 1 + \frac{1}{32} + \frac{1}{243} \ldots + \frac{1}{8^5} \approx 1.037$$

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b. How far off is S_8 from S? Use the remainder estimate for the integral test to determine this.

$$\left(\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx\right)$$

$$\frac{1}{4(9)^4} \le R_8 \le \frac{1}{4(8)^4}$$

$$0.000038 \le R_8 \le 0.000061$$

So S_8 is accurate to the nearest ten-thousandth because it is off by at most 0.000061.

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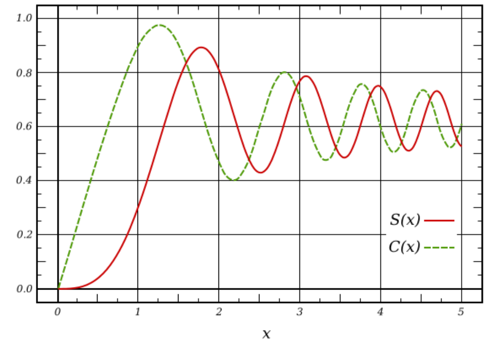
Fresnel Integrals $S(x)$ and $C(x)$ are two functions that are based on the integrals of two functions without elementary antiderivatives, $\sin(t^2)$ and $\cos(t^2)$. They are used in optics to calculate a diffraction pattern created by waves passing through an aperture or around an object.

$$S(x) = \int_0^x \sin(t^2)\ dt$$

$$C(x) = \int_0^x \cos(t^2)\ dt$$



5. Answer the following questions about Fresnel Integrals described above.

a. Write the Maclaurin series representations for $\sin(t^2)$ and $\cos(t^2)$ in expanded form to six terms.

$$\sin(t^2) = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \frac{t^{18}}{9!} - \frac{t^{22}}{11!} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n+2}}{(2n+1)!}$$

$$\cos(t^2) = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \frac{t^{16}}{8!} - \frac{t^{20}}{10!} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n}}{(2n)!}$$

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b. Integrate the Maclaurin series you found in part (a) indefinitely to obtain the following.

$$\int \sin(t^2)\ dt = \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} - \frac{t^{15}}{15 \cdot 7!} + \frac{t^{19}}{19 \cdot 9!} - \frac{t^{23}}{23 \cdot 11!} + \cdots + C = \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n+3}}{(4n+3)(2n+1)!} + C$$

$$\int \cos(t^2)\ dt = t - \frac{t^5}{5 \cdot 2!} + \frac{t^9}{9 \cdot 4!} - \frac{t^{13}}{13 \cdot 6!} + \frac{t^{17}}{17 \cdot 8!} - \frac{t^{21}}{21 \cdot 10!} + \cdots + C = \sum_{n=1}^{\infty} (-1)^n \frac{t^{4n+1}}{(4n+1)(2n)!} + C$$

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c. Approximate $S(2)$ and $C(2)$ by evaluating the Maclaurin series you obtained in part (b) from $0 \le t \le 2$.

$$S(2) = \int_0^2 \sin(t^2)\, dt \approx 0.805$$

$$C(2) = \int_0^2 \cos(t^2)\, dt \approx 0.461$$

6. Test the given series for convergence or divergence. Explain.

a. $$\sum_{n=1}^{\infty} \frac{\sqrt{4n^4 - 3n}}{5n^3 + 5}$$

$$b_n = \frac{\sqrt{4n^4}}{5n^3} = \frac{2n^2}{5n^3} = \frac{2}{5} \cdot \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{\sqrt{4n^4 - 3n}}{5n^3 + 5} \cdot \frac{5n}{2} = \lim_{n\to\infty} \frac{\sqrt{4n^6 - 3n^3}}{2n^3 + 2} = \lim_{n\to\infty} \frac{\sqrt{n^6}\sqrt{4 - \frac{3}{n^3}}}{n^3\left(2 + \frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{\sqrt{4 - \frac{3}{n^3}}}{\left(2 + \frac{2}{n^3}\right)} = \frac{\sqrt{4}}{2} = 1$$

So since $\sum \frac{1}{n}$ is a divergent p-series, this series also **diverges** by the Limit Comparison Test.

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b. $$\sum_{n=1}^{\infty} n!$$

$$\lim_{n\to\infty} n! = \infty$$

Therefore this series **diverges** by the Divergence Test.

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7. If the following series converges, find its exact sum. If not, explain why it diverges.

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$$

This series is geometric with $r = \frac{2}{3}$, therefore it **converges**, and it converges to

$$\sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^{n-1} = \frac{2}{1 - \frac{2}{3}} = 6$$